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Generalization of Gleason's Theorem

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The theorem of Gleason is proved without positivity and separability condition.

Let $\mathbb{K}$ denote the field of real or complex numbers or the skewfield of quaternions; let $\mathscr{H}$ be a (not necessarily separable) Hilbert space over $\mathbb{K}$. Let $\mathscr{L}(\mathscr{H})$ denote the orthocomplemented complete lattice of all closed subspaces of $\mathscr{H}$. A *Gleason measure* on $\mathscr{H}$ is a real functional m on $\mathscr{L}(\mathscr{H})$ which is σ-additive on all sequences of mutually orthogonal elements of $\mathscr{L}(\mathscr{H})$ and which is bounded [i.e., there exists $c \in \mathbb{R}_+$ with $|mM| \leqslant c$ for all $M \in \mathscr{L}(\mathscr{H})$]. Gleason measures characterize the states of physical systems (see Varadarajan, 1968, for instance). The theorem of Gleason (see Gleason, 1958) asserts that a positive Gleason measure m on a separable Hilbert space $\mathscr{H}$ with dimension $\geqslant 3$ is induced by a positive nuclear operator W_m on $\mathscr{H}$ via the formula $mM = \operatorname{Tr} W_m P^M$ [P^M is the projector on $M \in \mathscr{L}(\mathscr{H})$]. In Eilers and Horst (1975), and also Wepother (1974), it is proved that the assumption of separability, or, positivity, is superfluous. Here we show that both conditions are unnecessary by using an argument of Gleason's proof which gives new (and shortened) proofs also in the two special cases of Eilers and Horst, and Wepother.

> *Theorem:* Let $\mathscr{H}$ by a Hilbert space over $\mathbb{K}$ whose dimension is a not real-measurable[1] cardinal number $\neq 2$. Then each Gleason measure m on $\mathscr{H}$ is induced by a self-adjoint nuclear operator W_m on $\mathscr{H}$, which is uniquely determined by m, and the map $m \mapsto W_m$ is an isomorphic isometry from the Banach space $M(\mathscr{H})$ of Gleason measures on $\mathscr{H}$

[1] Presumably each cardinal number is not real-measurable; see below.

(with respect to the norm of total variation[2]) to the Banach space $T_r(\mathscr{H})$ of self-adjoint nuclear operators on $\mathscr{H}$ (with the trace norm) which preserves positivity [i.e., maps the positive cone of $M(\mathscr{H})$ on the positive cone of $T_r(\mathscr{H})$]. A cardinal I is said to be *real measurable* iff I is uncountable and there exists a positive measure $\mu \neq 0$ on the power set of I with $\mu(\{\alpha\}) = 0$ for each $\alpha \in I$. It is known that, on the basis of ZF (axioms of Zermelo–Fraenkel) and AC (axiom of choice) and GCH (generalized continuum hypothesis), the existence of real-measurable cardinals cannot be proved; on the contrary, much effort has been made to prove their nonexistence. In any case, real-measurable cardinals are, on the basis of ZF, AC, GCH,[3] much larger than all "good" cardinals (see Silver, 1971); for instance, they are inaccessible (see Ulam, 1930; for the definition of inaccessibility see Eilers and Horst).[4,5]

Corollary. Let $\mathscr{H}$ be a Hilbert space over $\mathbb{K}$ with nonmeasurable dimension difference of two positive Gleason measures m^+, m^- on $\mathscr{H}$. m^+ and m^- are uniquely determined by one of the following conditions:

(A) $\|m\| = \|m^+\| + \|m^-\|$ (Jordan decomposition)

(B) $\mathscr{H} = K_+ \oplus K_-$, and

$$m^+|_{K_+} = m|_{K_+}, m^-|_{K_-} = m^-|_{K_-}, m^+|_{K_-} = 0 = m^-|_{K_+}$$

(Hahn decomposition)

Remark. In the case of $\dim \mathscr{H} = 2$, a decomposition of a Gleason measure on $\mathscr{H}$ in two positive Gleason measures with the properties of (A) or of (B) is still possible, but uninteresting, because m^+ and m^- are determined by (A) or by (B) only in the trivial case where $m\mathbb{K}\varphi$ = constant for all $\varphi \neq 0$.

The proof of the theorem is given in several short steps:

[1] A Gleason measure m on a Hilbert space $\mathscr{H}$ with nonreal measurable dimension is totally additive.

[2] Defined by $\|m\| := \sup (\sum |mL_\alpha| : L_\alpha$ mutually orthogonal closed subspaces of $\mathscr{H})$.

[3] If the existence of a real-measurable cardinal is being allowable (as a new axiom of set theory consistent with ZF and AC), then the assumption of GCH gives no new inconsistency (see Silver, 1971).

[4] For the proof of the inaccessibility the "simple" continuum hypothesis is sufficient [see Drake (1974), Chap. 6, Theorems 1.3 and 1.7].

[5] If there exists a model of ZF, AC, GCH with a measurable cardinal I, then the cardinals less than I constitute (because of the inaccessibility of I) a model of ZF, AC, GCH without a measurable cardinal.

Proof. If $\mathscr{H}$ is separable nothing is to be proven. In the other case, let $\mathscr{B}$ be an orthonormal base (= ONB) of $\mathscr{H}$. By $\mu(\mathscr{B}') := m([\mathscr{B}'])$ for $\mathscr{B}' \subset \mathscr{B}$ a finite measure is defined on the power set of $\mathscr{B}$ ($[\mathscr{B}']$ denotes the closed subspace generated by $\mathscr{B}'$). Let μ_+, μ_- be the positive finite measures in the orthogonal decomposition of μ. As finite measures, μ_+ or μ_- have only at most denumerable sets D_+ or D_- of atoms. Because $\mathscr{B} - D_+$ and $\mathscr{B} - D_-$ have the same cardinality as $\mathscr{B}$, they are also nonreal measurable; therefore the restrictions of μ_+ or μ_- to the power sets of $\mathscr{B} - D_+$ or $\mathscr{B} - D_-$ vanish. Now the argument in the proof of Proposition 2 of Eilers and Horst (1975) completes the proof. ■

[2] The total variation of a Gleason measure m on a Hilbert space $\mathscr{H}$ is finite.

Proof. Let (L_α: $\alpha \in I$) be a family of mutually orthogonal closed subspaces of $\mathscr{H}$; let $\mathscr{B}_\alpha$ be ONB's of L_α and $\mathscr{B}$ be a completion of $\bigcup_\alpha \mathscr{B}_\alpha$ to an ONB of $\mathscr{H}$. Let μ be defined as in the proof of [1]. Then

$$\sum_\alpha |mL_\alpha| \leqslant \sum_\alpha |\mu\mathscr{B}_\alpha| \leqslant 2 \sup(|\mu\mathscr{B}'| : \mathscr{B}' \subset \mathscr{B}) \leqslant 2 \sup(|mL| : L \in \mathscr{L}(\mathscr{H}))$$

■

Remark. As a special case of [2], we have for each orthonormal system $\mathscr{B}$ of $\mathscr{H}$:

$$\sum_{\varphi \in \mathscr{B}} |m\mathbb{K}\varphi| \leqslant 2 \sup(|mL| : L \in \mathscr{L}(\mathscr{H}))$$

[3] Let m be a Gleason measure on a Hilbert space $\mathscr{H}$ of arbitrary dimension > 2. Let K, L be finite-dimensional subspaces of $\mathscr{H}$ with $K \subset L$ and $\dim K > 2$.[6] Then the restrictions of m to $\mathscr{L}(K)$ and $\mathscr{L}(L)$ are induced by nuclear operators W_K, W_L on K or L with $W_K = \pi W_L i$, where i denotes the inclusion $K \subset L$ and π the projection $L \to K$.

Proof. Because of [2], the definition

$$m_K M := mM + \|m\| \dim M \qquad (M \in \mathscr{L}(K)]$$

gives a positive Gleason measure on K which is induced, according to Gleason's theorem, by a positive nuclear operator W on K. Setting $W_K := W - \|m\| id_K$ we gain a self-adjoint nuclear operator on K with $mM = \operatorname{Tr} W_K P^M$ for $M \in \mathscr{L}(K)$. As a special case, for each normed $\varphi \in K$

$$(W_K\varphi, \varphi) = m\mathbb{K}\varphi \tag{1}$$

[6] This assumption is not necessary.

and on the other side: $((\pi W_L i)\varphi, \varphi) = \mathrm{Tr}_L W_L P^{\mathbb{K}\varphi} = m\mathbb{K}\varphi$; by polarization we conclude $W_K = \pi W_L i$. ■

[4] The first half of the theorem [compare with 3.4 of Gleason (1957)].

Proof. If $\dim \mathscr{H} = 1$ the theorem is trivial. In the other case, define $Q: \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ by $Q(\varphi, \psi) := (W_K\varphi, \psi)$, where $K = K(\varphi, \psi)$ is a finite-dimensional subspace of $\mathscr{H}$ with $\varphi, \psi \in K$. If L is another finite-dimensional subspace of $\mathscr{H}$ with $\varphi, \psi \in L$, then we have by [3]

$$(W_K\varphi, \psi) = (W_{[K\cup L]}\varphi, \psi) = (W_L\varphi, \psi)$$

therefore Q is well defined. The same argument shows that Q is sesquilinear [the meaning in the case of the quaternions is given in Varadarajan (1968), p. 61]. Obviously, $Q(\varphi, \varphi) \in \mathbb{R}$ for all $\varphi \in \mathscr{H}$ and

$$\sup(|Q(\varphi, \varphi)| : \|\varphi\| = 1) = \sup(|m\mathbb{K}\varphi| : \|\varphi\| = 1) \leqslant \|m\|$$

Therefore there exists a bounded self-adjoint operator W_m on $\mathscr{H}$ with $Q(\varphi, \psi) = (W_m\varphi, \psi)$. The remark after [2] shows together with equation (1):

$$\sup\left(\sum_{\varphi\in\mathscr{B}} |W_m\varphi, \varphi)| : \mathscr{B} \text{ ONB of } \mathscr{H}\right) \leqslant \|m\| \tag{2}$$

proving the nuclearity of W_m. Finally, equation (1) and the complete additivity of m show $mK = \mathrm{Tr}\, W_m P^K$ for each $K \in \mathscr{L}(\mathscr{H})$. ■

[5] The second half of the theorem.

Proof: Subjectivity of W follows from Sakai (1971), 1.15.3 and 1.13.2. Isometric mappings being injective we prove: $\|m\| = \mathrm{Tr}\,|W_m|$. First, by equation (2)

$$\mathrm{Tr}\,|W_m| = \sup\left(\sum_{\varphi\in\mathscr{B}} |(W_m\varphi, \varphi)| : \mathscr{B} \text{ ONB of } \mathscr{H}\right) \leqslant \|m\|$$

On the other side, let $(L_\alpha : \alpha \in I)$ be a family of mutually orthogonal closed subspaces of $\mathscr{H}$. Choose ONB's $\mathscr{B}_\alpha$ of L_α and complete $\bigcup_\alpha \mathscr{B}_\alpha$ to an ONB $\mathscr{B}$ of $\mathscr{H}$. Then, by [2]

$$\sum_\alpha |mL_\alpha| \leqslant \sum_\alpha \sum_{\varphi\in\mathscr{B}_\alpha} |m\mathbb{K}\varphi| \leqslant \sum_{\varphi\in\mathscr{B}} |(W_m\varphi, \varphi)| \leqslant \mathrm{Tr}\,|W_m|$$

$$\|m\| := \sup\left(\sum_\alpha |mL_\alpha|\right) \leqslant \mathrm{Tr}\,|W_m|$$

From $\|m\| = \mathrm{Tr}\,|W_m|$ it follows that the Gleason measures on $\mathscr{H}$ form a Banach space. Finally, the positivity of m implies the positivity of the forms $\varphi \mapsto (W_K\varphi, \varphi)$ and therefore the positivity of W_m. The converse is trivial. ■

Proof of the corollary. The existence of the decomposition and the uniqueness condition (A) follows from the second half of the theorem. For the condition (B), we take as K_+ the closed subspace generated by the eigenvectors of the positive eigenvalues of W_m (and as K_- the orthogonal complement). Conversely, let W^+, W^- be the positive nuclear operators on K_+, K_- associated with m^+, m^-. Then

$$\tilde{W}^+\varphi := \begin{cases} i_+ W^+\varphi \\ 0 \end{cases} \text{for} \quad \begin{matrix}\varphi \in K_+ \\ \varphi \in K_-\end{matrix}, \quad \tilde{W}^-\varphi := \begin{cases} i_- W^-\varphi \\ 0 \end{cases} \text{for} \quad \begin{matrix}\varphi \in K_- \\ \varphi \in K_+\end{matrix}$$

(where i_+, i_- denote the inclusions $K_+ \subset \mathscr{H}$, $K_- \subset \mathscr{H}$) are positive nuclear operators on $\mathscr{H}$ such that for $\varphi \in K_\pm$:

$$(W_m\varphi, \varphi) = m\mathbb{K} = \pm m^\pm \mathbb{K}\varphi = \pm(\tilde{W}^\pm\varphi, \varphi)$$

By orthogonal decomposition of φ as $\varphi_+ + \varphi_- + \varphi_0$ with $\varphi_\pm \in K_\pm \; \theta \, K_+ \cap K_-$, $\varphi_0 \in K_+ \cap K_-$ we arrive at $W_m = \tilde{W}^+ - \tilde{W}^-$.

By choosing ONB's of $K_\pm \; \theta \, K_+ \cap K_-$ and $K_+ \cap K_-$ we gain

$$\operatorname{Tr}|W_m| = \operatorname{Tr}\tilde{W}^+ + \operatorname{Tr}\tilde{W}^-$$

Therefore $\tilde{W}^+ = W_m^+$, $\tilde{W}^- = W_m^-$, giving the uniqueness of m^+ and m^-. ■

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